McKay equivalence for symplectic resolutions of singularities

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To the memory of Andrei Nikolaevich Tyurin.

Contents

1 Introduction.

Let K be an algebraically closed field of characteristic 0, let V be a finite-dimensional K-vector space equipped with a non-degenerate skew-symmetric form $\omega \in \Lambda^2(V^*)$, and let $\Gamma \subset Sp(V)$ be a finite subgroup. Suppose that we are given a resolution of singularities of the quotient variety $\pi : X \to V/\Gamma$ such that the symplectic form on the smooth part of V/Γ extends to a non-degenerate closed 2-form $\Omega \in H^0(\Omega_X^2)$. The aim of this paper is to prove

Theorem 1.1. *There exists an equivalence of $\mathcal{O}_V^\Gamma$-linear triangulated categories $D^b(\mathrm{Coh}(X)) \cong D^b(\mathrm{Coh}^\Gamma(V))$.*

A conjecture of this type was first made by M. Reid [R]; a more general statement was conjectured by A. Bondal and D. Orlov, [BOr, §5].

When $\dim(V) = 2$ such an equivalence is well-known, [KV], [GV]; in fact our argument relies on these results. Recently a similar statement was established by T. Bridgeland, A. King and M. Reid [BKR] for crepant resolutions of Gorenstein quotients of vector spaces of dimension 3. The approach of [BKR] is very elegant, but unfortunately, it works only in $\dim 3$. Our paper is an attempt to generalize at least the result, if not the approach to higher dimensions. We can only treat symplectic quotients, not arbitrary Gorenstein ones. Notice though that our additional assumption on the resolution is not restrictive — every crepant resolution X of a symplectic quotient singularity in fact carries a non-degenerate symplectic form (see e.g. [Ka1]).

The proof uses reduction to positive characteristic, and quantization of the symplectic variety X_k over a field k of characteristic $p > 0$. Our method is suggested by the results of [BMR].

The key ingredient of the proof is a quantization of X_k whose global sections coincide with the standard quantization of $H^0(\mathcal{O}_X) = H^0(V, \mathcal{O}_V)^\Gamma$. By this we mean a deformation of the structure sheaf $\mathcal{O}_X$ to a sheaf of non-commutative $\mathsf{k}[[h]]$-algebras $\mathcal{O}_h(X)$ such that the algebra of global sections $H^0(X, \mathcal{O}_h)$ is identified the subalgebra $\mathcal{W}^\Gamma \subset \mathcal{W}$ of Γ-invariant vectors in the (completed) Weyl algebra $\mathcal{W}$ of the vector space V.

It turns out that over the generic point of $\mathrm{Spec}(\mathsf{k}[[h]])$, the quantized algebra $\mathcal{O}_h$ is an Azumaya algebra over $X^{(1)}$, the Frobenius twist of X; it also yields an Azumaya algebra on $X_\mathsf{k}^{(1)}$. The category of modules over the latter is the category of coherent sheaves on some gerb over $X^{(1)}$.

One then argues that the above Azumaya algebra on $X^{(1)}$ is *derived affine*, i.e. the derived functor of global sections provides an equivalence between the derived category of sheaves of modules, and the derived category of modules over its global sections; this algebra of global sections is identified with the algebra W^Γ, where W is the reduction of the Weyl algebra at $h = 1$.

Furthermore, for large p we have a Morita equivalence between W^Γ and $\mathsf{W}\#\Gamma$, the smash-product of W and Γ. Thus we get an equivalence between $D^b(\mathcal{W}\#\Gamma\text{-mod}^{\mathrm{fg}})$ and the derived category of modules over the Azumaya algebra on $X^{(1)}$. The algebra W is an Azumaya algebra over $V^{(1)}$; thus, roughly

speaking, the latter equivalence differs from the desired one by a twist with a certain gerb. We then use the norm map on Brauer groups to pass from sheaves over a gerb to coherent sheaves on the underlying variety.

Then the equivalence over k of large positive characteristic is constructed; by a standard procedure we derive the desired statement over a field of characteristic zero.

Theorem 1.1 implies, more or less directly, that any crepant resolution X of the quotient V/Γ is the moduli space of Γ-equivariant Artinian sheaves on V satisfying some stability conditions (what is known nowadays as *G-constellations*). This is an important issue which deserves further study; however, spelling out precisely the stability conditions is a non-trivial problem, and we prefer to postpone it to a future paper. Among other things, in the case when $X = \mathrm{Hilb}^n(\mathbb{A}^2)$ is the Hilbert scheme of n points on the affine plane, this should give a computation-free proof of the so-called $n!$-Conjecture established recently by M. Haiman [H]. We also note that once one knows that X is a moduli space of G-constellations, one can prove Theorem 1.1 by the method of [BKR]; this is pretty useless, however, because to obtain a modular interpretation of an arbitrary smooth crepant resolution X one already needs Theorem 1.1. We would like to mention that [BKR, Corollary 1.3] is completely misleading in this respect, since it in fact assumes that the resolution is a G-Hilbert scheme (although this is not mentioned in the statement, and only appears as a sort of carry-over assumption from the previous page).

It is easy to see that the category of Γ-equivariant coherent sheaves on V is equivalent to the category of finitely-generated modules over a certain non-commutative algebra. Recently B. Keller [Ke] has given a definition of the so-called *Hochschild homology* groups of an abelian category. The Hochschild homology of the category of finitely-generated modules over an algebra coincides with the Hochschild homology of the algebra; in particular, the homology group HH_k is trivial for $k < 0$. On the other hand, the Hochschild homology group HH_k of the category of coherent sheaves on a smooth algebraic variety X over a characteristic 0 field is isomorphic to the sum of $H^{p-k}(X, \Omega_X^p)$. But Keller's definition is invariant with respect to functors which induce an equivalence of derived categories. Therefore Theorem 1.1 implies that $H^p(X, \Omega_X^q) = 0$ for $p > q$. In the symplectic case, this reduces to the more familiar $H^p(X, \Omega_X^q) = 0$, $p + q > \dim X$, which has been proved in [Ka2] (in fact, we use this vanishing to prove Theorem 1.1). However, in general, the required vanishing is very strange and quite strong. This raises some doubts as to whether Theorem 1.1 holds for crepant resolution of general Calabi-Yau quotient singularities V/Γ.

1.1 Notations. The pair $\langle V, \omega \rangle$ is a symplectic vector space, $\Gamma \subset Sp(V)$ is a finite subgroup, $\pi : X \to V/\Gamma$ is a fixed resolution of singularities such that ω extends to a symplectic form on the whole X, $\eta : V \to V/\Gamma$ is the projection map. The group algebra of Γ with coefficients in a ring R is denoted by $R[\Gamma]$. For an arbitrary k-algebra A equipped with a Γ-action, the *smash-product algebra* $A\#\Gamma$ coincides with $A[\Gamma]$ as an abelian group, and the multiplication is defined

by

$$(a_1 \cdot \gamma_1)(a_2 \cdot \gamma_2) = a_1 a_2^{\gamma_1} \cdot \gamma_1 \gamma_2, \qquad a_1, a_2 \in A, \gamma_1, \gamma_2 \in \Gamma.$$

The Weyl algebra W_h is defined by

$$\mathsf{W}_h = \mathsf{k}[h]\langle V^* \rangle / \left(xy - yx = h\omega^{-1}(x,y) \right);$$

the formal Weyl algebra $\mathcal{W}$ is the h-adic completion of W_h. We define open Γ-invariant subschemes $V_i \subset V$ by

$$V_i = \{ v \in V \mid \mathsf{dim}\, V^{Stab_\Gamma(v)} \leq 2i + \mathsf{dim}\, V^\Gamma \},$$

and we set $X_i = \pi^{-1}(V_i/\Gamma)$ (these will actually be used for $i = 0, 1$ only). We write $\mathcal{O}^p = \{ f^p \mid f \in \mathcal{O} \}$ for a commutative algebra $\mathcal{O}$ of characteristic p. For a scheme X over a field of characteristic p, $X^{(1)}$ is the Frobenius twist of X, and $\mathsf{Fr} : X \to X^{(1)}$ is the Frobenius morphism. In our applications X will be reduced; in this case $X^{(1)} = \langle X, \mathcal{O}_X^p \rangle$. Moreover, the base field will be perfect, so that the twist $X^{(1)}$ is isomorphic to X as an abstract scheme; however, it is often convenient to distinguish between the two notation-wise.

If $\mathcal{A}$ is an Azumaya algebra over a scheme X, we write $\mathrm{Coh}(X, \mathcal{A})$ for the category of coherent sheaves of $\mathcal{A}$-modules.

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Several years ago the first author gave a talk at the Shafarevich seminar at Steklov Math Institute in Moscow, where he described the results of Mirkovic and Rumynin and suggested, somewhat sceptically, some possible generalizations; the present paper is in large part the realization of the program sketched at that time. A.N. Tyurin was in the audience. He immediately believed in our vague hopes and enthusiastically supported the proposed research. This paper confirms that he was right. We were hoping to tell Andrei Nikolaevich about our results; sadly, this was not to be.

2 Almost exceptional objects.

The goal of this section is to show that Theorem 1.1 in large part follows from completely general homological arguments.

Definition 2.1. A nonzero object M of an abelian category is *almost exceptional* if $\mathrm{Ext}^i(M, M) = 0$ for $i > 0$, and the algebra $\mathrm{End}(M)$ has finite homological dimension.

Proposition 2.2. *Let Y be a smooth irreducible variety over a field k with trivial canonical class; assume that there exists a proper morphism from Y to an affine k-variety S. Let $\mathcal{A}$ be an Azumaya algebra over Y. If $\mathcal{E} \in \mathrm{Coh}(X, \mathcal{A})$ is an almost exceptional object, then the functor $\mathcal{F} \mapsto \mathrm{RHom}^{\bullet}(\mathcal{E}, \mathcal{F})$ from the derived category $D^b(\mathrm{Coh}(X, \mathcal{A}))$ to the derived category $D^b(\mathrm{End}(\mathcal{E})^{op}\text{-}\mathrm{mod}^{\mathrm{fg}})$ of finitely generated right $\mathrm{End}(\mathcal{E})$-modules is an equivalence.*

Proposition 2.2 applied to $\mathcal{A} = \mathcal{O}$ immediately shows that to prove Theorem 1.1, it suffices to prove the following.

Theorem 2.3. *There exists a vector bundle $\mathcal{E}$ on X such that*

(i) *the $\mathcal{O}_V^{\Gamma}$-algebra $\mathrm{End}(\mathcal{E})$ is isomorphic to the smash-product $\mathcal{O}_V \# \Gamma$,*

(ii) *we have $\mathrm{Ext}^i(\mathcal{E}, \mathcal{E}) = 0$ for $i > 0$.*

This is what we will do in the rest of the paper, after proving Proposition 2.2 in Subsection 2.2 (to eliminate confusion between right and left modules, we note right away that since $V \cong V^*$ as a Γ-module, we have $(\mathcal{O}_V \# \Gamma)^{op} \cong \mathcal{O}_V \# \Gamma$). Additionally, we will show in Subsection 2.3 that almost exceptional objects are remarkably rigid, so that essentially it suffices to construct such a bundle $\mathcal{E}$ after reduction to positive characteristic.

Remark 2.4. One can show that conditions on the vector bundle $\mathcal{E}$ in Theorem 2.3 are equivalent to the following conditions.

(a) $\mathcal{E}_{X_0} \cong \pi^* \eta_*(\mathcal{O})|_{X_0}$

(b) $\mathrm{Ext}^i(\mathcal{E}, \mathcal{E}) = 0$ for $i > 0$, and $\mathrm{End}(\mathcal{E}) \overset{\sim}{\longrightarrow} \mathrm{End}(\mathcal{E}|_{X_0})$.

2.1 Calabi-Yau categories. We recall some generalities from homological algebra. We refer to [BK] for details.

Let k be a field. A k-linear triangulated category is said to be *of finite type* if for any two objects $X, Y \in D$ the space $\mathrm{Ext}^{\bullet}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}(X, Y[i])$ is finite-dimensional. For such a category Bondal and Kapranov [BK] defined a *Serre functor* to be a pair consisting of a (covariant) auto-equivalence $S : D \to D$, and an isomorphism $\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(Y, S(X))^*$ fixed for all $X, Y \in D$, subject to certain compatibilities. If a Serre functor exists, it is unique up to a unique isomorphism. For example, if D is the bounded derived category of coherent sheaves on a smooth projective variety X over k, then $F \mapsto F \otimes K_X[\dim X]$ is a Serre functor for D (where K_X is the canonical line bundle).

We will need a slight generalization of this notion. Let $\mathcal{O}$ be a finitely generated commutative algebra over a field, and let D be an $\mathcal{O}$-linear triangulated category; moreover, assume given a functor $\mathsf{Hom} : D^{op} \times D \to D^b(\mathcal{O}\text{-}\mathrm{mod}^{\mathrm{fg}})$ and a functorial isomorphism $\mathrm{Hom}(X, Y) \cong H^0(\mathsf{Hom}(X, Y))$; we will call such a set of data *a strong $\mathcal{O}$-category.*

The triangulated category $D^b(\mathcal{O}\text{-}\mathrm{mod}^{\mathrm{fg}})$ is equipped with a canonical anti-auto-equivalence, namely, the Grothendieck-Serre duality [Ha]. We let $\mathbb{S}$ denote

this functor; thus $\mathbb{S} : \mathcal{F} \mapsto R\mathcal{H}om(\mathcal{F}, \mathcal{D})$, where $\mathcal{D}$ is the Grothendieck dualizing sheaf, and $\mathcal{H}om$ stands for the internal Hom.

Definition 2.5. A *Serre functor with respect to $\mathcal{O}$* (or just an *$\mathcal{O}$-Serre functor*) is a pair consisting of a (covariant) auto-equivalence $S : D \to D$, and a natural (functorial) isomorphism $\mathsf{Hom}(X, Y) \cong \mathbb{S}(\mathsf{Hom}(Y, S(X)))$ satisfying the compatibilities in [BK].

For example, if X is a smooth variety over k equipped with a projective morphism $\pi : X \to \operatorname{Spec}(R)$, then $D^b(\operatorname{Coh}(X))$ with $\mathsf{Hom}(\mathcal{F}, \mathcal{G}) = R\pi_*\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a strong R-category. The functor $\mathcal{F} \mapsto \mathcal{F} \otimes K_X[\mathsf{dim}\, X]$ is naturally a Serre functor with respect to R; this is true because Grothendieck-Serre duality commutes with proper direct images, and we have

$$\mathbb{S}(\mathcal{H}om(\mathcal{F}, \mathcal{G})) \cong \mathcal{H}om(\mathcal{G}, \mathcal{F} \otimes K_X[\mathsf{dim}\, X]).$$

The following generalization of this fact is straightforward.

Lemma 2.6. *Let X be a smooth variety over k equipped with a projective morphism $\pi : X \to \operatorname{Spec}(R)$ as above; and let $\mathcal{A}$ be an Azumaya algebra on X. Then $D^b(\operatorname{Coh}(X, \mathcal{A}))$ is naturally a strong R-category. The functor $\mathcal{F} \mapsto \mathcal{F} \otimes K_X[\mathsf{dim}\, X]$ is naturally a Serre functor with respect to R.* $\square$

A strong $\mathcal{O}$-category will be called *Calabi-Yau* if it admits a Serre functor with respect to $\mathcal{O}$ which is isomorphic to the shift functor $X \mapsto X[n]$ for some $n \in \mathbb{Z}$.

Application of the above notions to our situation is based on the following

Lemma 2.7. *Let $F : C \to D$ be a triangulated functor between non-zero triangulated categories. Assume that*

(i) *F has a left adjoint functor F' and the adjointness morphism $id \to F \circ F'$ is an isomorphism.*

(ii) *C is indecomposable, i.e. it can not be written as $C = C_1 \oplus C_2$ for nonzero triangulated categories C_1, C_2.*

(iii) *C admits a structure of a strong $\mathcal{O}$-category for some commutative algebra $\mathcal{O}$ of finite type over a field, so that C is Calabi-Yau with respect to $\mathcal{O}$.*

Then F is an equivalence.

Proof. Condition (i) implies that F' is a full embedding, and its (essential) image (which we denote by I) is a right admissible subcategory in C. Recall that this means that every object $\mathcal{F}$ of C fits into an exact triangle $\mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to \mathcal{F}_1[1]$ where $\mathcal{F}_1 \in I$, and $\mathcal{F}_2 \in I^\perp$; here

$$I^\perp = \{\mathcal{G} \in C \mid Hom(\mathcal{F}, \mathcal{G}) = 0 \quad \forall \mathcal{F} \in I\}$$

is the right orthogonal of C (to get such a triangle for a given $\mathcal{F}$ set $\mathcal{F}_1 = F'F(\mathcal{F})$, and let the arrow $\mathcal{F}_1 \to \mathcal{F}$ be the adjunction morphism; then complete it to an exact triangle). In this situation one says that C admits a semi-orthogonal decomposition. It follows from the definitions that if S is a Serre functor for C (relative to some commutative algebra $\mathcal{O}$ such that C is equipped with a strong $\mathcal{O}$-linear structure) then S^{-1} sends the right orthogonal to a full subcategory into the left orthogonal to the same subcategory. In particular, if C is Calabi-Yau relative to some algebra $\mathcal{O}$, then the left orthogonal to any triangulated subcategory coincides with the right orthogonal. Thus the above semi-orthogonal decomposition is actually orthogonal, i.e. we have $C = I \oplus I^\perp$; since $D \neq 0$ condition (ii) implies that $I^\perp = 0$, so $I = C$, and F, F' are equivalences. $\qquad\square$

Remark 2.8. A similar (more general) statement is proved and used in [BKR].

Another simple auxiliary fact is

Lemma 2.9. *Let X be a connected quasiprojective variety over a field, and let $\mathcal{A}$ be an Azumaya algebra on X. Then the category $D^b(\mathrm{Coh}(X, \mathcal{A}))$ is indecomposable.*

Proof. Assume that $D^b(\mathrm{Coh}(X, \mathcal{A})) = D_1 \oplus D_2$ is a decomposition invariant under the shift functor. Let P be an indecomposable summand of the free $\mathcal{A}$-module. Let L be an ample line bundle on X such that $H^0(L \otimes \mathcal{H}om_\mathcal{O}(P, P)) \neq 0$. For any $n \in \mathbb{Z}$ the $\mathcal{A}$-module $P \otimes L^{\otimes n}$ is indecomposable, hence belongs either to D_1 or to D_2. Moreover, all these modules belong to the same summand, because $Hom(P \otimes L^{\otimes n}, P \otimes L^{\otimes m}) \neq 0$ for $n \leq m$. If $\mathcal{F}$ is an object of the other summand, we have $Ext^\bullet(P \otimes L^{\otimes n}, \mathcal{F}) = 0$ for all n, which implies $\mathcal{F} = 0$. $\quad\square$

2.2 Proof of Proposition 2.2. We claim that the functor

$$F : \mathcal{F} \mapsto \mathrm{RHom}^\bullet(\mathcal{E}, \mathcal{F})$$

satisfies the conditions of Lemma 2.7. Indeed, the left adjoint functor is given by $F' : M \mapsto M \overset{L}{\otimes}_{\mathrm{End}(\mathcal{E})^{op}} \mathcal{F}$; it sends the bounded derived category in the bounded one because $\mathrm{End}(\mathcal{E})$ has finite homological dimension. Vanishing of $Ext^i(\mathcal{E}, \mathcal{E})$ implies that $\mathsf{id} \cong F' \circ F$. Thus (i) holds. Condition (ii) is provided by Lemma 2.9, and (iii) holds by Lemma 2.6. $\qquad\square$

2.3 Rigidity. Let R be a regular Noetherian ring with a maximal ideal $\mathfrak{m} \subset R$ and the residue field $k = R/\mathfrak{m}$. Assume given an algebraic variety X_R flat and smooth over R, and let $X = X_R \otimes_R k$ be the fiber of X over $\mathrm{Spec}\,k \in Spec R$.

Lemma 2.10. *Every almost exceptional vector bundle $\mathcal{E}$ on X extends uniquely to an almost exceptional vector bundle $\widehat{\mathcal{E}}$ on the formal completion $\mathfrak{X}$ of X_R in $X \subset X_R$.*

Proof. Filter R by the powers of $\mathfrak{m}$, and construct the extension step-by-step, by extending to $X_R \otimes_R (R/\mathfrak{m}^k)$ for all k in turn. By the standard deformation theory, at each step the obstructions to extending $\mathcal{E}$ lie in $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E})$, and different extensions are parametrized by a torsor over $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$. Since $\mathcal{E}$ is almost exceptional, both groups vanish. It remains to show that the extended bundle $\widehat{\mathcal{E}}$ is almost exceptional. Indeed, since $\mathrm{Ext}^i(\mathcal{E}, \mathcal{E}) = 0$ for all $i \geq 1$, we also have $\mathrm{Ext}^i(\widehat{\mathcal{E}}, \widehat{\mathcal{E}}) = 0$ for $i \geq 1$, $\mathrm{End}(\widehat{\mathcal{E}})$ is flat over R, and the natural map $\mathrm{End}(\widehat{\mathcal{E}})/\mathfrak{m} \to \mathrm{End}(\mathcal{E})$ is an isomorphism. To prove that the algebra $\mathrm{End}(\widehat{\mathcal{E}})$ has finite homological dimension, one computes the $\mathrm{Ext}^{\bullet}$-groups by using the spectral sequence associated to the $\mathfrak{m}$-adic filtration. $\square$

Our varieties will usually be non-compact, so that passing from $\mathfrak{X}$ to X_R is not automatic. To obtain global information, we will use $\mathbb{G}_m$-actions. Assume that $X_R = \mathrm{Proj}\,\mathcal{B}^{\bullet}$ is projective over an affine variety $\mathrm{Spec}\,\mathcal{B}^0$ over R.

Definition 2.11. We will say that an action of the group $\mathbb{G}_m$ on X_R is *positive-weight* if all weights of corresponding $\mathbb{G}_m$-action on $\mathcal{B}^{\bullet}$ are non-negative.

Lemma 2.12. *In the assumptions of Lemma 2.10, assume in addition that R is complete with respect to $\mathfrak{m}$-adic filtration, X_R is equipped with a positive-weight $\mathbb{G}_m$-actions and that $\mathcal{E}$ is $\mathbb{G}_m$-equivariant. Then $\mathcal{E}$ extends uniquely to a $\mathbb{G}_m$-equivariant vector bundle $\mathcal{E}_R$ on X_R.*

Sketch of a proof. Every extension of the multiplicative group by a unipotent group is split; it follows that the extension of $\mathcal{E}$ to $\mathfrak{X}$ provided by Lemma 2.10 can be equipped with a $\mathbb{G}_m$-equivariant structure. Under the positive-weight assumption, the completion functor is an equivalence of categories between $\mathbb{G}_m$-equivariant coherent sheaves on X_R and on $\mathfrak{X}$. To construct an inverse equivalence, one uses the Serre Theorem to interpret coherent sheaves as graded $\mathcal{B}^{\bullet}$-modules, and replaces a module with the the submodule of $\mathbb{G}_m$-finite sections. $\square$

In fact, it is not necessary to require that $\mathcal{E}$ is $\mathbb{G}_m$-equivariant – every almost exceptional vector bundle is automatically $\mathbb{G}_m$-equivariant with respect to any positive-weight $\mathbb{G}_m$-action. We do not prove this, because the proof is slightly technical; in our applications, $\mathbb{G}_m$-equivariant structure follows from a direct geometric argument (see Proposition 4.3).

3   Quantizations.

In this section we spell out some of the generalities on quantization of algebraic varieties in positive characteristic. We do not strive for generality, and only do the work needed for the proof of Theorem 1.1.

Till the end of the section all objects are assumed to be defined over a fixed field k.

3.1   Standard definitions. For a Poisson k-algebra $\langle \mathcal{O}, \{-,-\} \rangle$, its (formal) quantization is defined in the usual way; thus a quantization is an associative flat $k[[h]]$-algebra $\mathcal{O}_h$, complete separated in the topology generated by $h^i \mathcal{O}_h$, and equipped with an isomorphism $\mathcal{O}_h/h\mathcal{O}_h \cong \mathcal{O}$ such that the commutator in $\mathcal{O}_h$ equals $h\{-,-\} \mod h^2\mathcal{O}_h$.

If X is Poisson scheme, a quantization of X is a sheaf of $k[h]$-algebras $\mathcal{O}_h$ on X equipped with an isomorphism $\mathcal{O}_h/h\mathcal{O}_h \cong \mathcal{O}_X$ such that the algebra of sections $\mathcal{O}_h(U)$ for an affine open U is a quantization of $\mathcal{O}(U)$. One checks that quantizazion of an algebra defines a quantization of its localization (see e.g. [K, §2.1]), so that a quantization of a Poisson algebra A gives a quantization of the Poisson scheme $X = \operatorname{Spec} A$.

Below we will mostly be concerned with examples when X is a symplectic variety.

Example 3.1. For a smooth affine variety M over k, the algebra $D_h(M)$ of *asymptotic differential operators* on M is the h-completion of the algebra generated by $\mathcal{O}_M$ and $Vect(M)$ subject to the usual relations $f_1 \cdot f_2 = f_1 f_2$, $f \cdot \xi = f\xi$, $\xi \cdot f - f \cdot \xi = h\xi(f)$, $\xi_1\xi_2 - \xi_2\xi_1 = h[\xi_1, \xi_2]$. As usual one checks easily that $D_h(M)$ is a quantization of the sympectic variety T^*M.

Gluing the above construction, for any smooth variety M over k one obtains a sheaf $D_h(M)$ which is a quantization of T^*M.

Notice that when $\operatorname{char} k$ is positive, this construction is related to the so-called "crystalline" differential operators in the terminology of Mirkovic and Rumynin (PD differential operators in the terminology of [BO]) rather than to the more widely known Grothendieck differential operators (the latter contain divided powers of a vector field, while the former does not).

We will also need a graded version of quantizations. Assume that a Poisson scheme X is equipped with a $\mathbb{G}_m$-action such that the Poisson bracket has weight -2 (in other words, $\deg\{f,g\} = \deg f + \deg g - 2$ for any two homogeneous local functions f, g on X). We will say that that a quantization $\mathcal{O}_h$ of the scheme X is *graded* if it is equipped with a $\mathbb{G}_m$-action such that h has degree 2 and the isomorphism $\mathcal{O}_h/h \cong \mathcal{O}_X$ is $\mathbb{G}_m$-equivariant. E.g. the standard quantization $\mathcal{W}$ of a symplectic vector space is graded (see Example 3.6).

3.2   Quantizations as deformations. By its very definition, quantizations can be studied by deformation theory. Lately it has become fashionable to reduce all questions of deformation theory to solving the Maurer-Cartan equation in some differential graded Lie algebra. In characteristic 0, one can conceivably apply this method to quantizations and maybe even obtain something useful; however, this is irrelevant for us, since we are mostly interested in the positive characteristic case, where the differential graded Lie algebra formalism makes no sense. Thus we have to deal with deformations step-by-step, in the traditional standard way. This does not give much, but it is possible to prove one extension result which we will need.

Let X be a smooth variety. We recall that by the standard deformation theory, deformations of the structure sheaf $\mathcal{O}_X$ in the class of sheaves of associative algebras are controlled by the Hochschild cohomology groups $HH^i(X)$, $i = 1, 2, 3$ — that is, by the groups

$$\operatorname{Ext}^i_{X \times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta), \qquad i = 2, 3,$$

where $\Delta \subset X \times X$ is the diagonal. The group $HH^2(X)$ contains the deformation classes, and the group $HH^3(X)$ contains the obstructions.

Lemma 3.2. (i) *Let $U \subset X$ be an open subset, and assume that the restriction map*

$$HH^i(X) \to HH^i(U)$$

is bijective for $i = 1, 2$ and injective for $i = 3$. Then every quantization of U extends uniquely to a quantization of X.

(ii) *Assume that X is equipped with a $\mathbb{G}_m$-action which preserves $U \subset X$, so that the Poisson bracket $\{-, -\}$ in $\mathcal{O}_X$ has degree -2, and let $HH^\bullet_-(X) \subset HH^\bullet(X)$, $HH^\bullet_-(U) \subset HH^\bullet(U)$ be the subspaces of vectors of negative weight with respect to $\mathbb{G}_m$. Assume that the restriction map $HH^i_-(X) \to HH^i_-(U)$ is bijective for $i = 1, 2$ and injective for $i = 3$. Then every graded quantization of U extends uniquely to a graded quantization of X.*

Sketch of a proof. By an order-n deformation $\mathcal{O}_n$ of the structure sheaf $\mathcal{O}_X$ we will understand a sheaf of flat $\mathsf{k}[h]/h^{n+1}$-algebras on X equipped with an algebra isomorphism $\mathcal{O}_n/h \cong \mathcal{O}_X$. Fix a deformation $\mathcal{O}_{n-1}$ of order $n-1$. Then all order-n deformations $\mathcal{O}_n$ of $\mathcal{O}_X$ equipped with an isomorphism $\mathcal{O}_n/h^n \cong \mathcal{O}_{n-1}$ form a gerb bound by $HH^{\leq 3}(X)$ (this means that for some complex $C^\bullet$ with cohomology groups $HH^i(X)$, $i = 1, 2, 3$, and for some fixed element $c \in C^3$, all these deformations are parametrized by elements $b \in C^2$ with $d(b) = c$, and isomorphic deformations correspond to homologous elements b_1, b_2). Analogously, all such deformations of $\mathcal{O}_U$ form a gerb bound by $HH^{\leq 3}(U)$. To prove (i), note that the conditions of the Lemma insure that the restriction induces an equivalence of the corresponding gerbs, and apply induction on n. To prove (ii), note that in the $\mathbb{G}_m$-equivariant setting, since we have $\deg h = 2$, the relevant gerbs are bound by $HH^{\leq 3}(X)_{-2n} \subset HH^{\leq 3}(X)$, that is, the subspace of vectors of weight $-2n$. The conditions of the Lemma insure that the gerbs are equivalent at each step n. $\qquad\square$

3.3 Frobenius center. From now till the end of the section we assume that k is a perfect field k of characteristic $p > 0$. The Poisson scheme X is assume to be reduced.

The new feature of the positive characteristic theory is the presence of a huge center in the quantized algebra $\mathcal{O}_h$. In the situation of Example 3.1 it is closely related to the notion of *p-curvature* of a flat connection.

We start with an elementary observation that for a Poisson k-algebra $\mathcal{O}$ and $f \in \mathcal{O}$, the element $f^p \in \mathcal{O}$ lies in the center of the Lie algebra $\mathcal{O}$; thus the Poisson bracket is $\mathcal{O}^p$-linear, so that the Poisson bracket turns $\mathsf{Fr}_*(\mathcal{O})$ into a coherent sheaf of Lie algebras on the scheme $X^{(1)} = (X, \mathcal{O}_X^p)$.

Definition 3.3. A quantization $\mathcal{O}_h$ of a Poisson scheme X is called *Frobenius-constant* if the embedding $\mathcal{O}_X^p \hookrightarrow \mathcal{O}$ lifts to a morphism of sheaves of algebras from $\mathcal{O}_X^p$ to the center of $\mathcal{O}_h$

Frobenius-constant quantizations are more geometric than arbitrary ones. Namely, let $\mathcal{O}_h$ be a Frobenius-constant quantization of a Poisson scheme X. Then $\mathcal{O}_h$ is by definition a sheaf of $\mathcal{O}^p$-algebras, thus $\mathcal{O}_h$ defines a quasi-coherent sheaf on the formal scheme $\widehat{X}^{(1)}$, the completion of $X^{(1)} \times \operatorname{Spec} \mathsf{k}[h]$ at the central fiber $X^{(1)} \times \{0\}$. It is easy to see that this sheaf is locally free of rank $p^{\dim(X)}$. Its restriction to the special fiber of $\widehat{X}$ is identified with $Fr_*(\mathcal{O})$. As in Lemma 2.12, in the graded setting we have even more.

Lemma 3.4. *Let $X = \operatorname{Proj} \mathcal{B}^\bullet$ be a projective variety over $\operatorname{Spec} \mathcal{B}^0$, smooth over k and equipped with a $\mathbb{G}_m$-action and a Poisson bracket of degree -2. Assume that the $\mathbb{G}_m$-action on X is positive-weight. Then every graded Frobenius-constant quantization $\mathcal{O}_h$ of X is the completion of a unique $\mathbb{G}_m$-equivariant sheaf of algebras on the product $X^{(1)} \times \operatorname{Spec} \mathsf{k}[h]$.*

Proof. As in Lemma 2.12, under the positive weight assumption, the completion is an equivalence of tensor categories of $\mathbb{G}_m$-equivariant coherent sheaves on $X^{(1)} \times \operatorname{Spec} \mathsf{k}[h]$ and on $\widehat{X}^{(1)}$. $\square$

Frobenius-constant quantizations are local objects, both in Zariski and in étale topology. For an affine X, the functor of global sections provides an equivalence between Frobenius-constant quantizations of X and those of $\mathcal{O}_X$.

A motivation for the definition is provided by the next

Proposition 3.5. *If M is a smooth k-variety, then $D_h(M)$ is a Frobenius-constant quantization of T^*M.*

Sketch of a proof. We define a map from generators of $\mathcal{O}(T^*M)^p$ to $D_h(M)$. A function $f^p \in \mathcal{O}_M^p$ lifted from M is sent to $f^p \in D_h(M)$. A fiber-wise linear function on T^*M is a vector field on M; for such a function $\xi \in Vect(M)$ we set $\xi^p \mapsto \xi^p - h^{p-1}\xi^{[p]}$, where $\xi^{[p]}$ is the restricted power of a vector field ξ. The remaining part of the proof is a direct computation in local coordinates (cf. [BMR, §1.2]). $\square$

Example 3.6. The h-adic completion $\mathcal{W}$ of the Weyl algebra W_h (see Subsection 1.1) is a Frobenius-constant quantization of the symplectic space V. (This may be viewed as a particular case of Proposition 3.5 where M is a Lagrangian subspace in V). It is graded. More precisely, W_h carries a grading such that

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linear functions have degree 1, and $\deg(h) = 2$; it induces an action of the multiplicative group on $\mathcal{W}$, and W_h is identified with the space of $\mathbb{G}_m$ finite vectors in W_h.

Example 3.7. Let $\mathfrak{g}$ be a Lie algebra over k. Then the *asymptotic enveloping algebra* $U_h(\mathfrak{g})$ is the h-completion of the graded algebra $\mathsf{k}[h]\langle\mathfrak{g}\rangle/(xy - yx = h[x,y])$. If $\mathfrak{g}$ is a restricted Lie algebra, then $U_h(\mathfrak{g})$ is a Frobenius-constant quantization of the Poisson variety $\mathfrak{g}^*$; the Frobenius center is generated by $x^p - h^{p-1}x^{[p]}$, $x \in \mathfrak{g}$. (Notice that these central elements are homogeneous with respect to the natural grading, in which $deg(h) = 1$).

The following result is a generaliztion of a fundamental observation by Mirkovic and Rumynin.

Proposition 3.8. *Let $\mathcal{O}_h$ be a Frobenius-constant quantization of a symplectic variety X. Then the restriction $\mathcal{O}_h[h^{-1}]$ of $\mathcal{O}_h$ to the generic fiber of the formal scheme $\widehat{X}$ is an Azumaya algebra of rank $p^{\dim X}$.*

Proof. For a closed point $x \in X$, its Frobenius neighborhood $Frob(x) \subset X$ is by definition the spectrum of the fiber of the $\mathcal{O}_X^p$-algebra $\mathsf{Fr}_* \mathcal{O}_X$ at the point x. It is a Poisson scheme. It sufficies to check that for any closed point $x \in X$ and for any quantization A of the Poisson algebra of functions on $Frob(x)$, the localization $A(h^{-1})$ is an Azumaya algebra over $\mathsf{k}((h))$. Let Z be the center of $A(h^{-1})$. Then

$$\dim_{\mathsf{k}((h))} Z = \dim_{\mathsf{k}}(Z \cap A) \mod hA.$$

The right hand side is contained in the Poisson center of $\mathcal{O}_{Frob(x)}$ which is easily seen to be one dimensional. Thus $Z = \mathsf{k}((h))$. Assume now that $I \subset A(h^{-1})$ is a non-zero two-sided ideal. Then $(I \cap A) \mod h$ is a Poisson ideal in $\mathcal{O}_{Frob(x)}$. Since Hamiltonian vector fields generate the tangent space to X at x (this is immediate by a computation in local coordinates), it follows that a Poisson ideal is invariant under all derivations; on the other hand, one checks easily that $\mathcal{O}_{Frob(x)}$ has no proper nonzero ideals invariant under all derivations. Thus $(I \cap A) \mod h = A \mod h$, so that $I = A$ by Nakayama lemma. $\square$

4 Generalities on symplectic resolutions.

We return to the set-up of the introduction: V is a finite-dimensional symplectic vector space, $\Gamma \subset Sp(V)$ is a finite subgroup, $\pi : X \to Y = V/\Gamma$ is a smooth projective resolution, the symplectic form ω on $X_0 = V_0/\Gamma$ extends to a non-degenerate symplectic form on X.

We set $\mathcal{O}^\Gamma = \mathcal{O}(V)^\Gamma = H^0(X, \mathcal{O})$. Let the multiplicative group $\mathbb{G}_m$ act on the vector space V by dilations. This actions descends to an action on V/Γ, and then lifts to an action on $X \to V/\Gamma$ (see [Ka1]; the assumption $\mathsf{char}\,\mathsf{k} = 0$ adopted in [Ka1] is not essential). All the actions are positive-weight. It is also known [Ka1] that the map $X \to Y$ is necessarily semismall — that is,

$\dim X \times_Y X = \dim X$ (again, the proof in [Ka1] assumes characteristic 0, but it works without any changes in arbitrary characteristic).

In addition, we will assume that X satisfies the following:

- We have $H^p(X, \Omega^q) = 0$ when $p + q > \dim X$.

This is true in characteristic 0, see [Ka2]. It is probably also true in positive characteristic under some additional assumptions. We could not find these assumptions, unfortunately; therefore we simply impose ($\bullet$) as a standing assumption on the resolution X.

Example 4.1. If $\dim V = 2$, then every quotient V/Γ with $\Gamma \subset Sp(V) = SL(V)$ admits a unique resolution $X \to V/\Gamma$ satisfying all our assumptions. This situation is completely classic; we will give some more details in Subsection 5.1.

Recall that we have dense open subsets $V_0 \subset V_1 \subset \cdots \subset V$, and we denote $X_i = \pi^{-1}(V_i/\Gamma)$. We will need a particular étale covering of the open subset $X_1 \subset X$. Consider the connected components $H_\alpha \subset (V_1/\Gamma)$ of the complement $(V_1/\Gamma) \setminus (V_0/\Gamma)$. Every such H_α is a closed subscheme in V_1/Γ of pure codimension 2. There exists a linear subspace $V_\alpha \subset V$ of codimension 2 such that $H_\alpha = \eta(V_\alpha \cap V_1)$, and the subgroup $\Gamma_\alpha \subset \Gamma$ of elements in Γ stabilizing each vector in V_α is non-trivial.

Denote by $\Gamma'_\alpha \subset \Gamma$ the stabilizer of the subspace V_α; then Γ_α is a normal subgroup in Γ'_α. Consider V as a Γ_α-module. It splits as an orthogonal sum $V = W \oplus V_\alpha$, where $\dim W = 2$ (so that $\Gamma_\alpha \subset SL(W)$). The quotient group $N_\alpha = \Gamma'_\alpha/\Gamma_\alpha$ acts on W/Γ_α and on V_α; the second action is a symplectic action on a symplectic vector space. In keeping with our general notation, let $V_{\alpha,0} \subset V_\alpha$ be the open subset of vectors with trivial stabilizer in N_α.

The projection $\eta : V \to V/\Gamma$ induces a natural map

$$\eta_\alpha : (W/\Gamma_\alpha) \times V_{\alpha,0} \to ((W/\Gamma_\alpha) \times V_{\alpha,0}) / N_\alpha \to V/\Gamma.$$

The map η_α is étale outside of some closed subset $F_1 \subset (W/\Gamma_\alpha) \times V_{\alpha,0}$ which is disjoint from $\{0\} \times V_{\alpha,0}$. The preimage $\eta_\alpha^{-1}(H_\alpha)$ is the disjoint union of $(\{0\} \times V_{\alpha,0})/N_\alpha$ and some closed subset $F_2 \subset (W/\Gamma_\alpha) \times V_{\alpha,0}$.

Denote by $U_\alpha \subset (W/\Gamma_\alpha) \times V_{\alpha,0}$ the complement to the closed subset $F_1 \cup F_2$. Then U_α is a Zariski neighborhood of $\{0\} \times V_{\alpha,0} \subset (W/\Gamma_\alpha) \times V_{\alpha,0}$; the map $\eta_\alpha : U_\alpha \to V/\Gamma$ is étale, and $\eta_\alpha^{-1}(H_\alpha) = \{0\} \times V_{\alpha,0} \subset U_\alpha$. Moreover, η_α induces an isomorphism $\eta_\alpha : (\{0\} \times V_{\alpha,0})/N_\alpha \cong H_\alpha$.

The map $\eta_\alpha : (W/\Gamma_\alpha) \times V_{\alpha,0} \to V/\Gamma$ extends to a map

$$\rho_\alpha : (Y_\alpha \times V_{\alpha,0}) / N_\alpha \to X,$$

where $\pi_\alpha : Y_\alpha \to W/\Gamma_\alpha$ is the canonical crepant resolution of W/Γ_α (see e.g. [Ka1, Section 4]; since the resolution $Y_\alpha \to W/G_\alpha$ is canonical, the action of N_α on W/Γ_α lifts uniquely to an action on Y_α).

Denote by $X_\alpha \subset Y_\alpha \times V_{\alpha,0}$ the preimage $(\pi_\alpha \times \mathsf{id})^{-1}(U_\alpha)$ of the open subset $U_\alpha \subset (W/\Gamma_\alpha) \times V_{\alpha,0}$. The map ρ_α is étale on X_α; we have

(4.1) $$\rho_\alpha^{-1}\pi^{-1}(H_\alpha) = \pi_\alpha^{-1}(\{0\}) \times V_{\alpha,0},$$

and the map ρ_α induces an isomorphism

$$\rho_\alpha : \left(\pi_\alpha^{-1}(\{0\}) \times V_{\alpha,0}\right)/N_\alpha \to \pi^{-1}(H_\alpha) \subset X.$$

The sets X_α together with $X_0 \subset X_1$ form an étale covering of the subset $X_1 \subset X$. The following glueing lemma formalizes the situation.

Lemma 4.2. *The category of coherent sheaves $\mathcal{E}$ on X_1 is equivalent to the category of the following data:*

(i) *a coherent sheaf $\mathcal{E}_0$ on X_0,*

(ii) *for every component H_α of the complement $V_1/\Gamma \setminus V_0/\Gamma$, an extension $\mathcal{E}_\alpha$ of the sheaf $\rho_\alpha^*\mathcal{E}_0$ to an N_α-equivariant sheaf on X_α.*

Proof. Since X_0 and all the X_α together form an étale covering of X_1, it suffices to extend $\mathcal{E}_0$ to a sheaf on $\rho_\alpha(X_\alpha)$ for every X_α. To do this, it suffices to construct the descent data for the étale map $\rho_\alpha : X_\alpha \to X$. Let $X_{\alpha,0} \subset X_\alpha$ be the preimage $\rho_\alpha^{-1}(X_0)$. By (4.1), the product $X_\alpha \times_X X_\alpha$ is covered by two open subsets: $X_{\alpha,0} \times_X X_{\alpha,0}$ and $X_\alpha \times N_\alpha$. The descent data on the first subset are already given, since $\mathcal{E}_\alpha = \rho^{-1}\mathcal{E}_0$ on $X_{\alpha,0} \subset X_\alpha$. The descent data on the second subset are equivalent to the N_α-equivariant structure on $\mathcal{E}_\alpha$. $\qquad\square$

Proposition 4.3. (i) *Assume that Theorem 2.3 holds for some vector bundle $\mathcal{E}$. Then $\mathcal{E}$ carries a canonical $\mathbb{G}_m$-equivariant structure.*

(ii) *Let R be a regular complete local ring with residue field k, the maximal ideal $\mathfrak{m} \subset R$, and and fraction field K, and assume given a projective crepant resolution $\pi : X_R \to V_R/\Gamma$, where X_R is a scheme smooth over R. Assume that Theorem 2.3 holds for the special fiber X_k of the scheme X_R, and that for every one of the étale charts $X_\alpha \subset Y_\alpha \times V_{\alpha,0}$, the pullback $\rho^*\mathcal{E}$ extends to a vector bundle on the whole $Y_\alpha \times V_\alpha$. Then Theorem 2.3 holds for the generic fiber X_K.*

Proof. To prove (i), note that since we have $\mathcal{O}_V \subset \mathcal{O}_V\#\Gamma$, any object $\mathcal{E}$ satisfying the conditions of Theorem 2.3 comes equipped with a structure of an $\mathcal{O}_V$-module. In particular, on $X_0 \cong V_0/\Gamma \subset X$ we must have

$$\mathcal{E} \cong \eta_* \widetilde{\mathcal{E}}$$

for some sheaf $\widetilde{\mathcal{E}}$ on V_0. Since $\mathsf{rk}\,\mathcal{E} = |\Gamma|$ is equal to the degree of the étale map $\eta : V_0 \to V_0/\Gamma$, the sheaf $\widetilde{\mathcal{E}}$ must be a line bundle. Since the complement to $V_0 \subset V$ is of codimension ≥ 2, the Picard group $\mathrm{Pic}(V_0) = \mathrm{Pic}(V)$ is trivial;

therefore $\widetilde{E} = \mathcal{O}_V$ and $\mathcal{E} \cong \eta_*\mathcal{O}_V$ on $X_0 \subset X$. In particular, it carries a $\mathbb{G}_m$-equivariant structure on the open part $X_0 \subset X$.

Let $a : \mathbb{G}_m \times X \to X$ be the action map, and let $p : \mathbb{G}_m \times X \to X$ be the projection. Then an equivariant structure on $\mathcal{E}$ is given by an isomorphism $a^*\mathcal{E} \cong p^*\mathcal{E}$; we are given such an isomorphism on $\mathbb{G}_m \times X_0$, and we have to show that it extends to $\mathbb{G}_m \times X$. We will show that *every* isomorphism $a^*(\mathcal{E}) \cong p^*(\mathcal{E})$ defined on $\mathbb{G}_m \times X_0$ extends to $\mathbb{G}_m \times X$. Indeed, let $\mathfrak{X}$ be the formal neighborhood of $\{1\} \times X \subset \mathbb{G}_m \times X$, and let $\mathfrak{X}_0$ be the formal neighborhood of $\{1\} \times X_0 \subset \mathbb{G}_m \times X_0$; then it suffices to show that every isomorphism $a^*\mathcal{E} \cong p^*\mathcal{E}$ on $\mathfrak{X}_0$ extends to $\mathfrak{X}$. Since $\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) = 0$, the sheaf $\mathcal{E}$ extends uniquely to the (trivial) one-parameter deformation $\mathfrak{X}$ of the variety X. Therefore there exists at least some isomorphism $a^*\mathcal{E} \cong p^*\mathcal{E}$. But since $\mathrm{End}(\mathcal{E}) = \mathcal{O}_V \# \Gamma$ is the same as $\mathrm{End}(\mathcal{E}_{X_0})$, the Formal Function Theorem show that every section of $\mathcal{H}om(\mathcal{E},\mathcal{E})$ on $\mathfrak{X}_0$ extends to $\mathfrak{X}$; therefore any given isomorphism $a^*\mathcal{E} \cong p^*\mathcal{E}$ on $\mathfrak{X}_0$ indeed extends to $\mathfrak{X}$.

To prove (ii), assume that Theorem 2.3 holds for X_{k}. By (i) the corresponding vector bundle $\mathcal{E}$ on X_{k} is $\mathbb{G}_m$-equivariant. Therefore by Lemma 2.12 it extends to an almost exceptional vector bundle $\mathcal{E}_R$ on X_R.

We know that the algebra $\mathrm{End}_R(\mathcal{E}_R)$ is a flat deformation of the algebra $\mathrm{End}(\mathcal{E})$. It remains to prove that it in fact coincides with $Sym(V_R) \# \Gamma$.

Applying Lemma 2.12 to $\mathbb{G}_m$-equivariant sheaves on V_R, we see that it suffices to prove that

$$\mathrm{End}(\mathcal{E}_{\mathfrak{X}}) \cong \widehat{Sym(V_R)} \# \Gamma,$$

where $\mathfrak{X}$ is the formal completion of X_R along $X_{\mathsf{k}} \subset X_R$, and the completion on the right-hand side is taken with respect to the $\mathfrak{m}$-adic topology. Assume first that $V = W \oplus V'$ with $\mathsf{dim}\, W = 2$, and that the group Γ acts trivially on V', so that, in pariticlar, $X = X_1$. Then $\mathcal{E} \cong \eta_*\mathcal{O}_V$ on $X_0 = V_0/\Gamma \subset X_{\mathsf{k}}$, and all the $\mathbb{G}_m$-homogeneous elements in the group

$$\mathrm{Ext}^1(\mathcal{E}|_{X_0}, \mathcal{E}|_{X_0})$$

have strictly negative weight with respect to the $\mathbb{G}_m$-action. Therefore by standard deformation theory, the sheaf $\mathcal{E}$ on X_0 admits a *unique* $\mathbb{G}_m$-equivariant extension to a sheaf on $\mathfrak{X}_0$. This means that

$$\mathcal{E}_{\mathfrak{X}_0} \cong \widehat{\eta_*\mathcal{O}_V}$$

on $\mathfrak{X}_0 \subset \mathfrak{X}$. Now, the restriction to $\mathfrak{X}_0 \subset X$ induces a map

$$\mathrm{End}(\mathcal{E}_{\mathfrak{X}}) \to \mathrm{End}(\mathcal{E}_{\mathfrak{X}_0}) \cong \widehat{Sym(V_R)} \# \Gamma$$

of Noetherian topological R-algebras complete with respect to the $\mathfrak{m}$-adic topology, and this map is an isomorphism modulo $\mathfrak{m}$. Therefore it is an isomorphism.

In the general case, this argument does not work, because the extension group $\mathrm{Ext}^1(\mathcal{E}_{X_0}, \mathcal{E}_{X_0})$ becomes too large. However, we can consider the restriction $\mathcal{E}_{X_1}$ of the bundle $\mathcal{E}$ to $X_1 \subset X_{\mathsf{k}}$, and we claim that

$$\mathrm{Ext}^i(\mathcal{E}_{X_1}, \mathcal{E}_{X_1}) = 0$$

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for $i = 1, 2$. Indeed, consider the vector bundle $\mathcal{E}nd(\mathcal{E})$ on X_1. Theorem 2.3 (ii) implies that $R^k \pi_* \mathcal{E}nd(\mathcal{E}) = 0$ for $k \geq 1$, so that by Theorem 2.3 (i) we have

$$\mathrm{Ext}^i(\mathcal{E}, \mathcal{E}) \cong H^i(X_1, \mathcal{E}nd(\mathcal{E})) \cong H^i(V_1/\Gamma, \mathcal{O}_V \# \Gamma).$$

The algebra $\mathcal{O}_V \# \Gamma$ considered as a sheaf on V_1/Γ is the direct image of the trivial sheaf $\mathcal{O}_V \otimes \mathsf{k}[\Gamma]$ with respect to the quotient map $\eta : V_1 \to V_1/\Gamma$. Therefore

$$H^i(V_1/\Gamma, \mathcal{O}_V \# \Gamma) = H^i(V_1, \mathcal{O}_V) \otimes \mathsf{k}[\Gamma].$$

Since $\mathsf{codim}\, V \setminus V_1 \geq 4$, the right-hand side is indeed trivial for $i = 1, 2$.

Now, by the same argument as in the case $V = W \oplus V'$, it suffices to prove that $\mathrm{End}(\mathcal{E}_{\mathfrak{X}_1})$ coincides with the smash-product algebra $\widehat{Sym(V_R)} \# \Gamma$. Since $\mathrm{Ext}^1(\mathcal{E}_{X_1}, \mathcal{E}_{X_1}) = 0$, the vector bundle $\mathcal{E}$ on X_1 admits a unique extension to a vector bundle on $\mathfrak{X}_1$. Thus it suffices to construct at least *some* vector bundle $\mathcal{E}'$ on $\mathfrak{X}_1$ which extends $\mathcal{E}_{X_1}$ and has the correct endomorphism algebra. To do this, we apply Lemma 4.2, and deduce that it suffices to construct a N_α-equivariant extension $\mathcal{E}'_\alpha$ of the sheaf $\mathcal{E}_\alpha$ on each of the resolution $Y_\alpha \times V_\alpha \to V/\Gamma_\alpha$ corresponding to the étale charts X_α. But since $\Gamma_\alpha \subset \Gamma$ acts trivially on V_α, we have already proved that the extension $\mathcal{E}'_\alpha$ exists, and it is unique. In particular, it is N_α-equivariant. $\qquad\square$

5 Quantization of a symplectic resolution

Assume now that the base field k is a perfect field of characteristic $p > 0$; we will also assume that $p > \dim X$, $p > |\Gamma|$. The goal of this section is the following

Theorem 5.1. *In the above assumptions there exists a graded Frobenius-constant quantization $\mathcal{O}_h$ of X such that the $(\mathcal{O}^\Gamma)^p$-algebra $H^0(X, \mathcal{O}_h)$ of global section of $\mathcal{O}_h$ is isomorphic to the standard quantization $\mathcal{W}^\Gamma$; the isomorphism is compatible with the $\mathbb{G}_m$-actions.*

Lemma 5.2. *Assume that a graded Frobenius-constant quantization $\mathcal{O}_h$ of X is such that its restriction to the open stratum $X_0 \cong V_0/\Gamma$ is $\mathbb{G}_m$-equivariantly isomorphic to $\mathcal{W}|_{V_0^{(1)}}^\Gamma$. Then $\mathcal{O}_h$ satisfies the conditions of Theorem 5.1.*

Proof. Since the complement to V_0 has codimension at least 2, we have $\mathcal{W} = H^0(\mathcal{W}|_{V_0^{(1)}})$. Thus $H^0(\mathcal{O}_h)$ is a subalgebra in $\mathcal{W}^\Gamma = H^0(X_0, \mathcal{O}_h|_{X_0^{(1)}})$. Using vanishing of $H^1(X, \mathcal{O})$ we see that

$$H^0(\mathcal{O}_h)/hH^0(\mathcal{O}_h) \overset{\sim}{\longrightarrow} H^0(\mathcal{O}) = \mathcal{O}_V^\Gamma = \mathcal{W}^\Gamma/h\mathcal{W}^\Gamma,$$

which shows surjectivity of the map $H^0(\mathcal{O}_h) \to \mathcal{W}^\Gamma$. $\qquad\square$

The plan of the proof of the Theorem is as follows. By Lemma 5.2, it suffices to take the given quantization of X_0 and extend it to a quantization of X. In Subsection 5.2 we will extend this quantization to X_1, and then to the whole of X. Since $\mathsf{codim}(V_1 - V_0) = 2$, extension to X_1 reduces to the case $\dim(V) = 2$ which is treated separately in Subsection 5.1.

5.1 Dimension 2 **case.** Throughout this subsection we assume that the dimension $\dim V = 2$. Our aim here is

Proposition 5.3. *Theorem 5.1 holds if* $\dim(V) = 2$*. Moreover, the resolution* X *and quantization* $\mathcal{O}_h$ *are equivariant under the normalizer of* Γ *in* $Sp(V)$*.*

5.1.1 The resolution as a Hamiltonian reduction. Classification of the data $\langle V, \omega, \Gamma \rangle$ with $\dim(V) = 2$ is well-known (and goes back at least to Klein). They are in bijection with simply-laced Dynkin diagrams; the crepant resolution X of V/Γ is unique in each case. Recall a recent description of X as the Hilbert quotient of V by Γ due to Ito and Nakamura [IN] (see also the exposition in [N2]). Consider the subvariety of Γ-fixed points in the Hilbert scheme $\mathrm{Hilb}^n(V)$ of subschemes in V of length n, $n = |\Gamma|$, and let $\mathrm{Hilb}^\Gamma(V)$ be its connected component which contains elements of the form $\bigoplus_{\gamma \in \Gamma} \mathcal{O}_{\gamma(v)}$, $v \in V \setminus \{0\}$. Then we have $X \cong \mathrm{Hilb}^\Gamma(V)$.

The resolution X can also be obtained as a Hamiltonian reduction of an (algebraic) symplectic vector space by an action of a reductive group; this construction (in analytic set-up) was discovered by Kronheimer [Kr], and studied in works of Lusztig, Nakajima and others. Let us briefly describe a version of this construction (apparently due to Nakajima) based on the realization as the Hilbert quotient.

Let $R = \mathsf{k}[\Gamma]$ be the regular representation of the group Γ. The space $X = Hilb^\Gamma(V)$ is the moduli space of representations of $Sym(V^*)\#\Gamma$ which are isomorphic to R as a Γ-module, and are generated by a Γ-invariant vector. Let $\mathcal{R}$ denote the tautological vector bundle on X, i.e. the pushforward to X of the structure sheaf of the universal Γ-equivariant subscheme in $X \times V$.

Let M be the space parametrizing representations N of $T(V^*)\#\Gamma$ equipped with an isomorphism of Γ-modules $N \cong R$; here T denotes the tensor algebra. By definition, M is the vector space

$$M = \mathrm{Hom}_\Gamma(V^*, \mathrm{End}(R)) = \mathrm{Hom}_\Gamma(V^* \otimes R, R) = (V \otimes \mathrm{End}(R))^\Gamma$$

of Γ-equivariant maps from V^* to $\mathrm{End}(R)$.

(One can also describe M as the space of representations of the so-called "doubled quiver" corresponding to the affine Dynkin diagram of type A, D, E respectively; see e.g. [Lu]).

The vector space $\mathrm{End}(R)$ is equipped with the symmetric trace pairing $\mathsf{tr}(ab)$ and a Lie bracket $[-, -] : \mathrm{End}(R) \otimes \mathrm{End}(R) \to \mathrm{End}(R)$; both are Γ-invariant. The pairing tensored with the symplectic form ω on V gives a symplectic form Ω on $V \otimes \mathrm{End}(R)$; restricting it to the space M of Γ invariants we get a symplectic form Ω on M. The bracket tensored with the form ω gives a quadratic map $V \otimes \mathrm{End}(R) \to \mathrm{End}(R)$; let $\tilde{\mu} : M \to \mathrm{End}(R)$ denote its restriction to M.

The group $G = \mathrm{Aut}_\Gamma(R)/\mathbb{G}_m$ acts on M preserving Ω. Let $\mathfrak{g}$ be the Lie algebra of G. It is easy to check that the image of $\tilde{\mu}$ is contained in the subspace $\mathfrak{g}^* \subset (\mathrm{End}(R))^* = \mathrm{End}(R)$; thus we have a map $\mu : M \to \mathfrak{g}^*$. One readily checks that μ is the moment map for the action of G on M.

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The zero fiber $\mu^{-1}(0)$ parametrizes those representations of $T(V^*)\#\Gamma$ which factor through $Sym(V^*)\#\Gamma$; thus a point of $\mu^{-1}(0)$ defines a Γ-equivariant coherent sheaf on V. It turns out that one of the GIT quotients of $\mu^{-1}(0)$ by G coincides with X. Namely, consider the splitting $\iota : G \to \mathrm{Aut}(R)$ of the projection $\mathrm{Aut}(R) \to G$ which identifies G with the subgroup of automorphisms which are trivial on the 1-dimensional space of Γ-invariant vectors. Define a character $\chi : G \to \mathbb{G}_m$ by $\chi(g) = \det(\iota(g))$. Then a point $x \in \mu^{-1}(0)$ is χ-stable iff the corresponding Γ-equivariant coherent sheaf on V is generated by a Γ-invariant section (see e.g. [N1], cf. also [Ki]). Thus the symplectic variety $X = \mathrm{Hilb}^\Gamma(V) = \mu^{-1}(0)//_\chi G$ is the Hamiltonian reduction of M by G.

In these terms one can also describe the tautological vector bundle $\mathcal{R}$. To this end, consider the trivial vector bundle on M with fiber R. The splitting ι can be used to equip it with a G-equivariant structure: we define an action of G on R by pulling-back the tautological action of G on R via ι; we then let G act on $R \otimes \mathcal{O}$ diagonally.

The restriction of the resulting G-equivariant bundle to $\mu^{-1}(0)$ descends to a vector bundle on the quotient. The resulting vector bundle on X is identified with $\mathcal{R}$.

As usual in the Hamiltonian reduction picture, we can also do the steps in reverse order: first take the GIT quotient $M//_\chi G$, then X is realized as a closed subscheme in $M//_\chi G$.

To sum up, we have

Proposition 5.4.   (i) *Let $M^{ss} \subset M$ be the open subscheme of representations generated by a Γ-invariant vector. Then the action of G on M^{ss} is free, and the geometric quotient M^{ss}/G exists. The subvariety*

$$\left(\mu^{-1}(0) \cap M^{ss}\right)/G \subset M^{ss}/G$$

is isomorphic to X, and the reduction of the form Ω equals ω.

(ii) *Equip the trivial vector bundle $R \otimes \mathcal{O}$ with the diagonal G-action. Then the descent of $R \otimes \mathcal{O}_{M^{ss}}$ to M^{ss}/G restricts to $\mathcal{R}$ on $X \subset M^{ss}/G$.* $\square$

5.1.2   Quantum version. To obtain quantizations by Hamiltonian reduction, we introduce the following

Definition 5.5. Let X be a symplectic algebraic variety over a field k of positive characteristic. Let G be an algebraic group acting on X. A Frobenius-constant quantization $\mathcal{O}_h$ of X is *Frobenius G-constant* if G acts on $\mathcal{O}_h$ so that the action of G on the central subalgebra $\mathcal{O}_X^p[[h]] \subset \mathcal{O}_h$ fixes h and coincides with the natural G-action on $\mathcal{O}_X^p \subset \mathcal{O}_X^p[[h]]$.

We will abuse terminology by saying "a map from a vector space W to a sheaf $\mathcal{F}$" instead of "a map from W to the space of global sections of $\mathcal{F}$".

Definition 5.6. Let X, G be as above, let $\mathfrak{g}$ be the Lie algebra of the group G, and let $\mathcal{O}_h(X)$ be a Frobenius G-constant quantization of X. A *quantum moment map* is a $\mathsf{k}[[h]]$-algebra homomorphism $\mu : U_h(\mathfrak{g}) \to \mathcal{O}_h(X)$ such that

(i) The restriction of μ to the central subalgebra $Sym(\mathfrak{g}^{(1)})[[h]]$ maps the subalgebra $Sym(\mathfrak{g}^{(1)}) \subset Sym(\mathfrak{g}^{(1)})[[h]]$ into $\mathcal{O}(X^{(1)}) \subset \mathcal{O}(X^{(1)})[[h]]$.

(ii) For every $\xi \in \mathfrak{g}$ and every local section s of $\mathcal{O}_h$ we have $\mu(\xi)s - s\mu(\xi) = h\xi(s)$, where $\xi(s)$ is the action of the Lie algebra $\mathfrak{g}$ on $\mathcal{O}_h$ induced by the action of G.

One checks immediately that by virtue of (ii), the induced map

$$\mu_0 : Sym(\mathfrak{g}^{(1)}) \to \mathcal{O}_X^p$$

whose existense is guaranteed by (i) is a moment map for the G-action on X.

Example 5.7. Let $X = V$ be a symplectic vector space, and let $G = Sp(V)$. Then we have a quantum moment map sending $x \in \mathfrak{sp}(V) \cong Sym^2(V) \subset V \otimes V$ into the corresponding element of $\mathcal{W}$.

If $G \subset Sp(V)$ is an algebraic subgroup, then we get a quantum moment map $\mu : \mathfrak{g} \to \mathcal{W}$ by restricting the above map to the Lie subalgebra $\mathfrak{g} \subset \mathfrak{sp}(V)$.

Notice that the induced homomorphism $U_h(\mathfrak{g}) \to \mathcal{W}$ (which we also denote by μ) is related to the $\mathbb{G}_m$-action by the formula $\mu(tx) = t^2\mu(x)$.

Proposition 5.8. *Let M be a smooth symplectic manifold, let G be a group acting on M, let $\mathcal{O}_h$ be a Frobenius G-constant quantization of X, and assume given a quantum moment map $\mu : \mathfrak{g} \to \mathcal{O}_h$. Moreover, assume given a G-invariant open subset $U \subset X$ such that the action of G on U is free, and the geometric quotient U/G exists; thus the projection $\rho : U \to U/G$ identifies U with the total space of a principal G-bundle on U/G. Then the sheaf*

$$(\rho_*\mathcal{O}_h)^G \subset \rho_*\mathcal{O}_h$$

is a Frobenius-constant quantization of the Poisson manifold U/G, and its quotient

$$\mathcal{O}_h(Z) = (\rho_*\mathcal{O}_h)^G / \langle \rho_*\mathcal{O}_h\mu(\mathfrak{g})\rho_*\mathcal{O}_h\rangle^G$$

is a Frobenius-constant quantization of the Hamiltonian quotient

$$Z = \left(\mu_0^{-1}(0) \cap U\right)/G \subset U/G.$$

Moreover, if $\mathcal{F}$ is a locally projective G-equivariant sheaf of left $\mathcal{O}_h$-modules on X, then the sheaf $(\rho_*\mathcal{F})^G / I(\rho_*\mathcal{F})^G$ is a locally projective sheaf of $\mathcal{O}_h(Z)$-modules on Z. $\qquad\square$

Sketch of a proof. If $S \subset U/G$ is an affine open subset, then, since the action of G on $\rho^{-1}(S)$ is free, the algebra of functions on $\rho^{-1}(S)$ is an injective object

in the category of algebraic G-modules. This proves the first statement. On the other hand, since the action is free, $\mathcal{O}(\rho^{-1}(S))$ is flat over $Sym(\mathfrak{g})$, and $\mu(\mathfrak{g})\mathcal{O}(\rho^{-1}(S))$ is an injective G-module. This implies an isomorphism

$$(5.1) \qquad \mathcal{O}_Z = \rho_*(\mathcal{O})^G/(\mu(\mathfrak{g})\rho_*(\mathcal{O})^G) \overset{\sim}{\longrightarrow} (\rho_*(\mathcal{O})/\mu(\mathfrak{g})\rho_*(\mathcal{O}))^G.$$

Also, $\mathcal{O}_h(\rho^{-1}(S))$ is flat over $U_h(\mathfrak{g})$, hence $(\rho_*(\mathcal{O}_h)/\mu(\mathfrak{g})\rho_*(\mathcal{O}_h))^G$ is an h-flat deformation of $\mathcal{O}(Z)$. The isomorphism (5.1) implies

$$\mathcal{O}_h(Z) = \rho_*(\mathcal{O}_h)^G/(\mu(\mathfrak{g})\rho_*(\mathcal{O}_h)^G) \overset{\sim}{\longrightarrow} (\rho_*(\mathcal{O}_h)/\mu(\mathfrak{g})\rho_*(\mathcal{O}_h))^G.$$

As noted above, the right hand side here is an h-flat deformation of $\mathcal{O}(Z)$, and the left hand side carries associative multiplication. It is clear that the commutator map in the resulting sheaf of algebras is compatible with the Poisson bracket; thus $\mathcal{O}_h(Z)$ is indeed a quantization of $\mathcal{O}_Z$. The proof of the statement about the Hamiltonian reduction of a $\mathcal{O}_h$-module is parallel to the proof of the statement about the reduction of the sheaf of algebras $\mathcal{O}_h$. $\qquad\square$

5.1.3 Quantizing the resolution. We now return to the situation of Proposition 5.3. Combining Proposition 5.4 with Proposition 5.8, we obtain a Frobenius-constant quantization $\mathcal{O}_h$ of the resolution $X \to V/G$. It is clear from the construction that the quantization $\mathcal{O}_h$ is graded. By Lemma 5.2, to prove Proposition 5.3, it remains to show that $\mathcal{O}_h$ coincides with the standard quantization on $V_0 \subset V$.

Consider the sheaf $\mathcal{R}_h$ of $\mathcal{O}_h$-modules on $X^{(1)}$ obtained as the Hamitonian reduction of the free module $R \otimes \mathcal{O}_h(M)$. Then by Proposition 5.8 together with Proposition 5.4 (ii), we have $\mathcal{R}_h/h\mathcal{R}_h \cong \mathcal{R}$.

Moreover, consider the restriction of the sheaf $\mathcal{R}_h$ to the open subset $X_0 = V_0/\Gamma \subset X$. Then we claim that there exists a natural isomorphism

$$\mathcal{R}_h \cong \mathcal{O}_h \otimes_{\mathcal{O}_{X_0}^p} \mathcal{O}_{V_0}^p$$

of sheaves on X_0. Indeed, the definition of $\mathcal{R}_h$ yields an action of $\mathcal{O}(V^{(1)})[[h]] = H^0(X^{(1)}, \mathcal{R}^{(1)})[[h]]$ on $\mathcal{R}_h$ commuting with the action of $\mathcal{O}_h$. This action, in turn, yields a map of sheaves

$$\mathcal{O}_{V_0}^p \otimes_{\mathcal{O}_{V/\Gamma}^p} \mathcal{O}_h \to \mathcal{R}_h.$$

It is readily seen to induce an isomorphism modulo h; since both sheaves are h-flat it is an isomorphism.

To prove Proposition 5.3, it remains to apply the following

Lemma 5.9. *Assume that $\mathcal{O}_h$ is a $\mathbb{G}_m$-equivariant Frobenius-constant quantization of X such that the sheaf*

$$\mathcal{O}_h|_{X_0} \otimes_{\mathcal{O}_{X_0}^{(1)}} \mathcal{O}_{V_0^{(1)}}$$

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of $\mathcal{O}_h$-modules on $X_0 = V_0/\Gamma$ extends to a locally projective $\mathcal{O}_h$-module $\mathcal{R}_h$ on X satisfying $\mathcal{R}_h/h\mathcal{R}_h \cong \mathcal{R}$.

Then the restriction of the quantization $\mathcal{O}_h$ to the open subset $X_0^{(1)} = V_0^{(1)}/\Gamma$ is isomorphic to $\mathcal{W}|_{V_0^{(1)}}^\Gamma$; the isomorphism is compatible with the $\mathbb{G}_m$-actions.

Proof. It suffices to construct a Γ-equivariant isomorphism of graded $(\mathcal{O}^\Gamma)^p$-algebras

$$(5.2) \qquad H^0\left(\mathcal{O}_h \otimes_{\mathcal{O}_{X_0}^{(1)}} \mathcal{O}_{V_0^{(1)}}\right) \cong \mathcal{W}.$$

Consider the subspace $V^* \subset \mathcal{O}(V)$ of linear functions on V. It follows e.g. from [KV] that $H^i(X, \mathcal{R}) = 0$ for $i > 0$. Hence the map

$$H^0(\mathcal{R}_h) = H^0(\mathcal{O}_h \otimes_{\mathcal{O}_{X_0}^{(1)}} \mathcal{O}_{V_0^{(1)}}) \to H^0(\mathcal{R}) = \mathcal{O}(V)$$

is surjective. In particular, we can factor the embedding $V \subset \mathcal{O}(V)$ through a map $\iota : V^* \to H^0(\mathcal{R}_h)$. Moreover, we can choose a $\mathbb{G}_m$-equivariant map, so that $\mathsf{Im}(\iota)$ is contained in the space of elements of degree 1 with respect to the $\mathbb{G}_m$-action. Then for any $x, y \in V^*$, the element $xy - yx$ equals $h\omega(x, y)$ modulo h^2, and has degree 2 with respect to the $\mathbb{G}_m$-action. This clearly implies $xy - yx = h\omega(x, y)$. Thus we get a Γ-equivariant algebra homomorphism from the completed Weyl algebra $\mathcal{W}$ to

$$H^0(\mathcal{O}_h \otimes_{\mathcal{O}_{X_0}^{(1)}} \mathcal{O}_{V_0^{(1)}}).$$

Since it induces an isomorphism on the associated graded spaces with respect to the h-adic filtrations, it is itself an isomorphism. $\qquad\square$

5.2   The general case. Return to the situation of Theorem 5.1, the general case (no assumption on $\mathsf{dim}\, V$). Recall notations V_1, X_1 from Subsection 1.1.

Proposition 5.10. *The standard quantization $\mathcal{W}^\Gamma$ of $X_0 = V_0/\Gamma$ extends to a graded Frobenius-constant quantization $\mathcal{O}_h$ of X_1.*

Proof. A Frobenius-constant quantization of X is a coherent sheaf on $X(1)$, so that Lemma 4.2 applies. Thus it suffices to extends the given quantization of $X_0 = V_0/\Gamma \subset X$ to a quantization of each of the étale charts X_α. Propositon 5.3 applied to Y_α gives a graded Frobenius-constant N_α-equivariant quantization of Y_α. Taking its (h-adically completed) tensor product with the standard quantization of the vector space V_α, we obtain by restriction a graded Frobenius-constant N_α-equivariant quantization of $X_\alpha \subset Y_\alpha \times V_\alpha$. $\qquad\square$

To finish the proof of Theorem 5.1, it remains to show that the Frobenius-constant quantization $\mathcal{O}_h$ of $X_1 \subset X$ extends to a Frobenius-constant quantization of the whole X. It suffices to prove the following.

Proposition 5.11. *Every graded Frobenius-constant quantization $\mathcal{O}_h$ of $X_1 \subset X$ extends uniquely to a graded Frobenius-constant quantization of X.*

Proof. First of all observe that it is enough to extend $\mathcal{O}_h$ to a deformation of $\mathcal{O}(X)$ to a sheaf of associative algebras: then the action of $\mathbb{G}_m$, and the embedding of $\mathcal{O}^p$ into the center of $\mathcal{O}_h$ extend uniquely from X_1, because $\mathsf{codim}(X \setminus X_1) \geq 2$.

We will use Lemma 3.2. The Hochschild cohomology of smooth varieties can be computed by the Hochschild-Kostant-Rosenberg spectral sequence, whose E_2-term is given by

$$H^p(X, \Lambda^q \mathcal{T}(X)),$$

the cohomology of degree p with coefficients in the bundle $\Lambda^q \mathcal{T}(X)$ of polyvector fields of degree q on X. Then Lemma 3.2 reduces the Proposition to the following cohomological statement.

Lemma 5.12. *Consider the restriction map*

$$\sigma : H^p(X, \Lambda^q \mathcal{T}(X)) \to H^p(X_1, \Lambda^q \mathcal{T}(X_1))$$

on the components of negative weight. Then it is an isomorphism if $p+q = 1, 2$, and it is injective if $p + q = 3$.

Proof. Since the sheaves of polyvector fields are locally free, while the complement $X \setminus X_1 \subset X$ is of codimension at least 2, the map σ is a bijection for $p = 0$ and an injection for $p = 1$ (for all weights).

By our assumption ($\bullet$), we have

$$H^p(X, \Omega_X^q) = 0$$

when $p + q > \dim X$. Since X is symplectic, we have $\Omega_X^p \cong \Lambda^{\dim X - p} \mathcal{T}(X)$, so that this implies

$$H^p(X, \Lambda^q \mathcal{T}(X)) = 0$$

when $p > q$. For $q = 0$ this gives $H^p(X, \mathcal{O}_X) = 0$, $p \geq 1$, and $H^p(X_1, \mathcal{O}_X) = H^p(V_1/\Gamma, \mathcal{O}_{V/\Gamma}) = H^p(V_1, \mathcal{O}_V)^\Gamma$. Since $\mathsf{codim} \, V \setminus V_1 \geq 4$, this implies that $H^p(X_1, \mathcal{O}_X) = 0$ for $p = 1, 2$. For $q = 1$ we obtain $H^2(X, \mathcal{T}(X)) = 0$, so that the map σ is tautologically injective for $p = 2$, $q = 1$.

It remains to consider the sum of components of negative weight in the group $H^1(X, \mathcal{T}(X))$. We can identify $\mathcal{T}(X) \cong \Omega_X^1$ by means of the symplectic form; since the symplectic form has weight 2, we are reduced to studying the sum of components of non-positive weights in $H^1(X, \Omega_X^1)$. Moreover, since the $\mathbb{G}_m$-action on X is positive-weight, we only have to analyze the $\mathbb{G}_m$-invariant part in $H^1(X, \Omega_X^1)$. We already know that the map σ is injective on $H^1(X, \Omega_X^1)$. Since $\mathsf{codim} \, X \setminus X_1 \geq 2$, every line bundle on X_1 extends to the whole of X. Therefore to finish the proof, it suffices to prove the following.

Lemma 5.13. *The group $H^1(X_1, \Omega_X^1)^{\mathbb{G}_m}$ is generated by Chern classes of line bundles on X_1.*

Proof. The sheaf $\pi_*\Omega^1_X$ on V_1/Γ coincides with the Γ-invariant part in the sheaf $\eta_*\Omega^1_V$, where $\eta : V \to V/\Gamma$ is the quotient map. Since $\mathsf{codim}\, V \setminus V_1 \geq 4$, we have $H^i(V_1/\Gamma, \pi_*\Omega^1(X)) = 0$ for $i = 1, 2$. Therefore

$$H^1(X_1, \Omega^1_X) = H^0(V_1/\Gamma, R^1\pi_*\Omega^1_X).$$

The sheaf $R^1\pi_*\Omega^1_X$ is concentrated on the complement $V_1/\Gamma \setminus V_0/\Gamma$, which is a disjoint union of the components H_α. Therefore on V_1/Γ we have

$$R^1\pi_*\Omega^1_X = \bigoplus_\alpha \left(\eta_{\alpha,*}\eta^*_\alpha R_1\pi_*\Omega^1_X \right)^{N_\alpha},$$

where $\eta_\alpha : U_\alpha \to V_1/\Gamma$ are the étale charts constructed in Section 4. By base change we have

$$\eta^*_\alpha R^1\pi_*\Omega^1_X \cong R^1\pi_*\Omega^1_{X_\alpha}.$$

Recall that X_α is an open dense subset in $\overline{X}_\alpha = Y_\alpha \times V_\alpha$. The sheaf $\Omega^1_{\overline{X}_\alpha}$ on $\overline{X}_\alpha$ decomposes as a direct sum

$$\left(\Omega^1_{Y_\alpha} \boxtimes \mathcal{O}_{V_\alpha} \right) \oplus \left(\mathcal{O}_{Y_\alpha} \boxtimes \Omega^1_{V_\alpha} \right),$$

and since $H^1(Y_\alpha, \mathcal{O}_{Y_\alpha}) = 0$, we have

$$R^1(\pi_\alpha \times id)_*(\mathcal{O}_{Y_\alpha} \boxtimes \Omega^1_{V_\alpha}) = 0$$

by the projection formula. Thus

$$R^1\pi_*\Omega^1_{X_\alpha} \cong R^1\pi_{\alpha,*}\Omega^1_{Y_\alpha} \boxtimes \mathcal{O}_{V_\alpha}.$$

We conclude that

$$H^0(V_1/\Gamma, R^1\pi_*\Omega^1_{X_1}) \cong \bigoplus_\alpha \left(H^1(Y_\alpha, \Omega^1_{Y_\alpha}) \otimes H^0(U_\alpha, \mathcal{O}_{V_{\alpha,0}}) \right)^{N_\alpha},$$

and since the complement $V_\alpha \setminus V_{\alpha,0}$ is of codimension at least 2, we have

$$H^0(U_\alpha, \mathcal{O}_{V_{\alpha,0}}) \cong \mathcal{O}_{V_\alpha}$$

for each α. It remains to prove that the group of weight-0 elements in

(5.3) $$\left(H^1(Y_\alpha, \Omega^1_{Y_\alpha}) \otimes \mathcal{O}_{V_\alpha} \right)^{N_\alpha}$$

is generated by Chern classes of N_α-equivariant line bundles on $\overline{X}_\alpha$. But it is easy to see that $H^1(Y_\alpha, \Omega^1_{Y_\alpha})$ is generated by the Chern classes of the exceptional curve $E = \pi^{-1}_\alpha(0) \subset Y_\alpha$, and $\mathbb{G}_m$ acts trivially on this group. On the other hand, the only functions of non-positive weight on V_α are constants. Thus every N_α-invariant element of weight 0 in the group (5.3) is a linear combination of Chern classes of divisors of the form $D \times V$, where D is a N_α-invariant divisor supported on E. Every such divisor $D \times V$ corresponds to a divisor on X. $\square$

6 Equivalences

From now on, subindices will be used to denote the ring of scalars; thus for an R-algebra $A = A_R$, the ring $A \otimes_R R'$ will be denoted by $A_{R'}$.

6.1 Assumptions. Fix an algebraically closed field K of characteristic zero, and the set of data $\langle V, \omega, \Gamma, X \rangle$ defined over K – V is a finite-dimesnional vector space over K, ω is a symplectic form on V, $\Gamma \subset Sp(V)$ is a finite subgroup, and $X \to V/\Gamma$ is a projective smooth crepant resolution of the quotient V/Γ. Let $R \subset K$ be a $\mathbb{Z}$-algebra of finite type such that $\langle V, \omega, \Gamma, X \rangle$ are defined over R; we assume $R \cong \mathbb{A}_R^n$, ω_R is symplectic; $\Gamma \subset Sp(n, R)$, X is smooth over R, and $\pi : X \to V/\Gamma$ is proper (below we will just say "defined over R" leaving the natural conditions implicit). Then the Weyl algebra W_h is also defined over R. It is clear that such an R exists.

By [Ka1] the map $X \to V/\Gamma$ is semismall. By [Ka2], this implies that $H^p(X, \Omega_X^q) = 0$ when $p + q > \dim X$. Therefore localizing R if necessary, we may assume that the reduction X_k to any closed point $\operatorname{Spec} \mathsf{k} \in \operatorname{Spec} R$ of positive characteristic satisfies the assumption ($\bullet$) in the beginning of Section 5 (to check that the bad loci are Zariski-closed algebraic subvarieties, one uses the positive-weight $\mathbb{G}_m$-action). Localizing R even further, we can insure that in fact all the assumptions of Section 5 are satisfied for every X_k.

Denote $\mathsf{W} = \mathsf{W}_h/(h-1)\mathsf{W}_h$.

Lemma 6.1. *There exists a dense affine open subset $\operatorname{Spec}(R') \subset \operatorname{Spec}(R)$ such that the bimodule $\mathsf{W}_{R'}$ gives a Morita equivalence*

$$\mathsf{W}_{R'}^\Gamma \sim (\mathsf{W}\#\Gamma)_{R'}.$$

Proof. We can assume that $|\Gamma|^{-1} \in R$. Set

$$e = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \in R[\Gamma] \subset (\mathsf{W}\#\Gamma).$$

Then e is an idempotent, $\mathsf{W}_{R'}^\Gamma = e\left((\mathsf{W}\#\Gamma)_{R'}\right)e$, and it is well-known that it suffcies to check that e generates the unit two-sided ideal in $(\mathsf{W}\#\Gamma)_{R'}$. The algebra W_K is simple; it is easy to deduce that the algebra $(\mathsf{W}_\mathsf{K}\#\Gamma)$ is also simple; so that e generates the unit ideal in $(\mathsf{W}\#\Gamma)_\mathsf{K}$. Hence it also generates the unit ideal in $(\mathsf{W}\#\Gamma)_{R'}$ for some $R' \subset K$ of finite type over R. $\square$

6.2 Equivalence for the Azumaya algebra in positive characteristic. Take R' satisfying Lemma 6.1, and fix a closed point $\operatorname{Spec} \mathsf{k} \in \operatorname{Spec} R'$. Let $X = X_\mathsf{k}$, $V = V_\mathsf{k}$. Consider the graded Frobenius-constant quantization $\mathcal{O}_h$ of X constructed in Theorem 5.1. Extend it to a sheaf O_h on $X \times \operatorname{Spec} \mathsf{k}[h, h^{-1}]$ by Lemma 3.4. Set $\mathsf{O} = \mathsf{O}_h/(h-1)\mathsf{O}_h$; thus, O is a locally free sheaf on $X^{(1)}$.

Lemma 6.2. *We have $H^i(\mathsf{O}) = 0$ for $i > 0$, and $H^0(\mathsf{O}) = \mathsf{W}^\Gamma$ as a $\mathcal{O}(V^{(1)})^\Gamma$-algebra.*

Proof. This statement immediately follows from the constructionl however, it deserves to be stated as a Lemma, since it is the crucial point in the proof of Theorem 1.1. □

By Proposition 3.8, the algebra O is an Azumaya algebra of rank $p^{\dim X}$.

Theorem 6.3. *We have equivalences of derived categories*

$$D^b\left(\mathrm{Coh}(X,\mathsf{O})\right) \cong D^b\left(\mathsf{W}_\mathsf{k}^\Gamma\text{-mod}^\mathsf{fg}\right) \cong D^b\left(\mathsf{W}\#\Gamma_\mathsf{k}\text{-mod}^\mathsf{fg}\right),$$

where the first equivalence is given by the derived functor of global sections, and the second one is provided by Lemma 6.1.

Proof. The second equivalence is given by Lemma 6.1. It remains to show that the derived functor of global sections provides an equivalence

$$D^b\left(\mathrm{Coh}(X,\mathsf{O})\right) \cong D^b\left(\mathsf{W}_\mathsf{k}^\Gamma\text{-mod}^\mathsf{fg}\right).$$

We will check that the free rank 1 module over O is an almost exceptional object in $\mathrm{Coh}(X^{(1)},\mathsf{O})$; then the desired statement follows from Proposition 2.2.

The algebra W has finite homological dimension, since its associated graded algebra with respect to the natural filtration is the symmetric algebra $Sym(V^*)$, which has finite homological dimension. Since $\mathsf{char}\,\mathsf{k}$ does not divide $|\Gamma|$, the algebra $\mathsf{W}\#\Gamma$ also has finite homological dimension; by the equivalence of Lemma 6.1, the same is true for the algebra W^Γ. Since we have $H^i(X,\mathcal{O})$ for $i \geq 1$, the object $\mathsf{O} \in \mathrm{Coh}(X_\mathsf{k}^{(1)},\mathsf{O})$ is indeed almost exceptional. □

6.3 Untwisting. Keep the notation of the last Subsection. We will now eliminate the Azumaya algebras.

Proposition 6.4. *There exists an Azumaya algebra $\mathcal{A}$ on $V^{(1)}/\Gamma$ whose restriction to $V_0^{(1)}/\Gamma$ is equivalent to $\mathsf{W}^\Gamma|_{V_0^{(1)}/\Gamma}$.*

The Proposition follows directly from the next

Lemma 6.5. *Let S be an affine scheme acted upon by a finite group Γ with $|\Gamma|$ invertible on S. Let $p : S \to S/\Gamma$ be the projection to the categorical quotient. Let $l \in \mathbb{Z}_{>0}$ be prime to $|\Gamma|$. Then p^* induces an isomorphism $Br(S/\Gamma)[l] \widetilde{\longrightarrow} Br(S)[l]^\Gamma$ where $[l]$ denotes the l-torsion.*

Proof. By a Theorem of O. Gabber [G], for any affine scheme X we have $Br(X)[l] = H^2(X_{et},\mathbb{G}_m)[l]$. For any finite morhism p, the higher direct images in the étale topology $R^i p_*(\mathbb{G}_m)$, $i \geq 1$ are trivial; indeed, the stalk of $R^i p_*(\mathbb{G}_m)$ at a geometric point x is the i-th cohomology group of the étale sheaf $\mathcal{O}^*$ on the spectrum of some commutative ring R which is finite over a strictly Henselian ring (see [M, Theorem III.1.15]); this cohomology group vanishes since R is itself strictly Henselian by [M, Corollary I.4.3].

We have the norm map $Nm : p_*(\mathbb{G}_{mS}) \to \mathbb{G}_{mS/\Gamma}$. It induces a map

$$Nm : Br(S)[l] = H^2\left((S/\Gamma)_{et}, p_*(\mathbb{G}_m)\right)[l] \to H^2\left((S/\Gamma)_{et}, \mathbb{G}_m\right)[l] = Br(S/\Gamma)[l].$$

We have $Nm \circ p^*(c) = |\Gamma| \cdot c$ and $p^* \circ Nm(c) = \sum_{\gamma \in \Gamma} \gamma^*(c)$. The statement follows. $\qquad\square$

Remark 6.6. This Proposition is suggested by a result of [BMR], where an analogue of the above Azumaya algebra $\mathcal{A}$ appears. More precisely, in *loc. cit.* one contructs explicitly an Azumaya algebra on the nilpotent cone in a simple Lie algebra $\mathfrak{g}$ over a field of characteristic p. This algebra is obtained from the enveloping $U(\mathfrak{g})$ by reduction at the "singular" character of the central subalgebra $Sym(\mathfrak{t})^W$; the singular central character comes from the singular weight $-\rho$ (here $\mathfrak{t}$ is the maximal torus of $\mathfrak{g}$, and W is the Weyl group).

Theorem 6.7. *Assume that the data $\langle V, \omega, \Gamma, X \rangle$ over the positive characteristic residue field $\mathsf{k} = R'/\mathfrak{m}$ satisfy the assumptions of Theorem 6.3. Then they also satisfy Theorem 2.3 (and consequently, Theorem 1.1).*

Proof. Let $\mathcal{A}$ be the Azumaya algebra constructed in Proposition 6.4. We claim that there exist equivalences

$$(6.1) \qquad \mathrm{Coh}(X^{(1)}) \cong \mathrm{Coh}(X^{(1)}, \mathsf{O} \otimes \pi^*(\mathcal{A}^{op}));$$

$$(6.2) \qquad D^b\left(\mathrm{Coh}\left(X^{(1)}, \mathsf{O} \otimes_{\mathcal{O}_{V^{(1)}}} \mathcal{A}^{op}\right)\right) \cong D^b\left(\mathsf{W}_\mathsf{k}^\Gamma \otimes_{\mathcal{O}_{V^{(1)}}^\Gamma} \mathcal{A}^{op}\text{-mod}^{\mathrm{fg}}\right);$$

$$(6.3) \qquad \mathsf{W}_\mathsf{k}^\Gamma \otimes_{\mathcal{O}_{V^{(1)}}^\Gamma} \mathcal{A}^{op}\text{-mod}^{\mathrm{fg}} \cong \left(\left(\mathsf{W}_\mathsf{k} \otimes_{\mathcal{O}_{V^{(1)}}} \eta^*(\mathcal{A}^{op})\right) \#\Gamma\right)\text{-mod}^{\mathrm{fg}};$$

$$(6.4) \qquad \mathrm{Coh}^\Gamma(V^{(1)}) \cong \left(\left(\mathsf{W}_\mathsf{k} \otimes_{\mathcal{O}_{V^{(1)}}} \eta^*(\mathcal{A}^{op})\right) \#\Gamma\right)\text{-mod}^{\mathrm{fg}}.$$

Here (6.1), and (6.4) follow once we show that the Azumaya algebra $\mathsf{O} \otimes \pi^*(\mathcal{A}^{op})$ (respectively, $\mathsf{W}_\mathsf{k} \otimes_{\mathcal{O}_{V^{(1)}}} \eta^*(\mathcal{A}^{op})$) appearing in the right-hand side is split (respectively, admits a Γ equivariant splitting). In each case they split on an open subvariety by the characterizing property of $\mathcal{A}$; hence they split on the whole variety by [M, Corollary IV.2.6] (Brauer group of a regular irreducible scheme injects into the Brauer group of its generic point); the Γ equivariant structure on the splitting bundle for $\mathsf{W}_\mathsf{k} \otimes_{\mathcal{O}_{V^{(1)}}} \eta^*(\mathcal{A}^{op})$ extends from V_0 to V since $\mathsf{codim}(V \setminus V_0) \geq 2$.

The equivalence (6.3) is parallel to the one in Lemma 6.1, and follows from the fact that the idempotent $e = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma}$ generates the unit two-sided ideal in $(\mathsf{W}_\mathsf{k} \otimes \eta^*(\mathcal{A}^{op})) \#\Gamma$ since it does so in $\mathsf{W}_\mathsf{k} \#\Gamma$.

Finally, the equivalence (6.2) is parallel to the one in Theorem 6.3. Namely, by the projection formula we get

$$H^0\left(\mathsf{O} \otimes_{\mathcal{O}_{V^{(1)}}^\Gamma} \mathcal{A}^{op}\right) = \mathsf{W}_\mathsf{k}^\Gamma \otimes_{\mathcal{O}_{V^{(1)}}^\Gamma} \mathcal{A}^{op}.$$

The projection formula also gives vanishing of the higher cohomology groups of the sheaf in the left hand side of the last equality. The algebra in the right-hand side has finite homological dimension by (6.3). Thus (6.2) follows from Proposition 2.2.

Composing the equivalences (6.1)–(6.4) we get the equivalence of the categories of coherent sheaves; now for $\mathcal{E}$ we can take the image of the structure sheaf $\mathcal{O}_V \# \Gamma \in \mathrm{Coh}^\Gamma(V)$ under this equivalence.

It remains to check this $\mathcal{E}$ is a vector bundle; the equivalence of categories then shows that it satisfies the conditions of Theorem 2.3. We use the following more explicit description of $\mathcal{E}$. Let $\mathcal{B}$ be the $\mathsf{O} \otimes \pi^*(\mathcal{A}^{op})$ module providing the Morita equivalence between O and $\pi^*(\mathcal{A})$; and let B be the $\mathsf{W}_h \otimes \eta^*(\mathcal{A}^{op})$ module providing the Morita equivalence between W_h and $\eta^*(\mathcal{A})$. It follows from the definition that the above equivalence sends the coherent sheaf $\mathcal{B} \in Coh(X^{(1)})$ to $B \otimes \mathsf{k}[\Gamma] \in Coh^\Gamma(V^{(1)})$. The vector bundle B on $V^{(1)}$ is trivial by the Suslin-Quillen Theorem. Thus $\mathcal{O}\#\Gamma$ is a direct summand in $B \otimes \mathsf{k}[\Gamma]$, so $\mathcal{E}$ is a direct summand in $\mathcal{B}$; in particular, it is a vector bundle. $\qquad\square$

Remark 6.8. One can somewhat simplify the above explicit description of $\mathcal{E}$ if one is only interested in the restriction of $\mathcal{E}$ to the formal neighborhood of a fiber of π. There the Azumaya algebra O splits, and indecomposable summands of $\mathcal{E}$ are indecomposable summands of a splitting vector bundle for O.

6.4 Characteristic zero. We can now finish the proof of Theorem 1.1. It suffices to prove Theorem 2.3.

Lemma 6.9. *Let $\mathsf{k}_1 \subset \mathsf{k}_2$ be fields (not necessarily algebraically closed). Assume V, ω, Γ, X are defined over k_1. If the statements of Theorem 2.3 holds for V, ω, Γ, X then it also holds for V_{k_2}, ω_{k_2}, Γ, X_{k_2}. Conversely, if it holds for V_{k_2}, ω_{k_2}, Γ, X_{k_2}, then it also holds for $V_{\mathsf{k}_1'}$, $\omega_{\mathsf{k}_1'}$, Γ, $X_{\mathsf{k}_1'}$ for some finite extension k_1' of k_1.*

Proof. The first statement is obvious. To prove the second one, consider a finite-type k_1-algebra $R \subset \mathsf{k}_2$ such that both the vector bundle $\mathcal{E}_{\mathsf{k}_2}$ whose existence is asserted in Theorem 2.3, and the isomorphism $\mathrm{End}(\mathcal{E}) \cong \mathcal{O}_V \# \Gamma$ are defined over R. Then k_2 can be taken to be the residue field of any closed point of $\mathrm{Spec}(R)$. $\qquad\square$

Proof of Theorem 2.3. Return to the notation of Subsection 6.1; replace R with R' provided by Lemma 6.1. Take an arbitrary closed point $\mathrm{Spec}\,\mathsf{k} \in \mathrm{Spec}\,R$ such that R is regular in $\mathrm{Spec}\,\mathsf{k}$, and the residue field $\mathsf{k} = R/\mathfrak{m}$ has positive characteristic. Let $\widetilde{R}$ be the completion of R with respect to the $\mathfrak{m}$-adic topology, and let K' be its fraction field. By Theorem 6.7, Theorem 2.3 holds for X_k. Moreover, by construction the vector bundle $\mathcal{E}$ on X_k satisfies the assumption of Proposition 4.3. Therefore Theorem 2.3 also holds for $X_{\mathsf{K}'}$. Applying Lemma 6.9, we see that Theorem 2.3 holds over a finite extension F of the fraction field of the

ring R. Since F is isomorphic to a subfield of K, we see that the Theorem holds in the original situation. $\qquad\square$

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